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1984 J. Phys. A: Math. Gen. 17 375

(http://iopscience.iop.org/0305-4470/17/2/023)

J. Phys. A: Math. Gen. **17** (1984) 375–383. Printed in Great Britain

Quantum stochastics: the passage from a relativistic to a non-relativistic path integral

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Received 2 June 1983

Abstract. Feynman's path integral for the one-dimensional Dirac particle uses paths for which $\Delta x \sim \Delta t$. For non-relativistic path integrals, typical paths satisfy $(\Delta x)^2 \sim \Delta t$ (as in Brownian motion). We demonstrate the consistency of these two stochastic schemes and show how the relativistic formalism contains within itself the scale of the transition regime, namely $\Delta x \sim \hbar/mc$, the Compton wavelength.

1. Introduction

The Dirac equation for a particle of mass m in one space dimension may be written

$$-\mathrm{i}\sigma_z\,\partial\psi/\partial x - m\sigma_x\psi = \mathrm{i}\,\partial\psi/\partial t \tag{1}$$

where $\hbar = c = 1$, σ_x and σ_z are Pauli spin matrices and ψ has two components. Feynman observed that the retarded propagator for this equation could be obtained from the following limiting process:

$$K_{\beta\alpha}(b, t_b; a, t_a) = \lim_{N\to\infty} \sum_{R\geqslant 0} \Phi_{\beta\alpha}(R)(\mathrm{i}\varepsilon m)^R \tag{2}$$

where $\varepsilon = (t_b - t_a)/N$, α and β take the values 'right', and 'left' and $\Phi_{\beta\alpha}(R)$ is the number of paths of exactly N steps, each of length ε (or $c\varepsilon$), that start at a in the direction α, end at b in the direction β and reverse direction R times (see Feynman and Hibbs (1965) for pictures). The indices α and β refer to the components of ψ. In this sum over paths, the space and time steps are of the *same* size in the sense that they scale in the same way with N.

For non-relativistic physics the propagator of a free particle of mass m in one dimension takes the form

$$K^{(\mathrm{NR})}(b, t_b; a, t_a) = \lim_{N\to\infty}\left(\frac{-m\mathrm{i}}{2\pi\hbar\varepsilon}\right)^{\frac{1}{2}(N-1)} \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \exp\left(\frac{\mathrm{i}m\,\Sigma\,(x_{i+1}-x_i)^2}{2\varepsilon}\right). \tag{3}$$

Now 'all' paths are contemplated, but it is a well known feature of the path integral that the bulk of the contributors to the sum satisfy $(\Delta x)^2 \sim \Delta t$. The term 'bulk of the contributors' achieves a more precise meaning in the corresponding Brownian motion problem (let $\mathrm{i}\to -1$ in (3)) in that $(\Delta x)^2 \sim \Delta t$ almost everywhere in Wiener measure.

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0305-4470/84/020375+09$02.25

Thus for a variety of reasons it makes sense to assign a diffusion constant

$$(\Delta x)^2/\Delta t = 1/m \qquad (= \hbar/m) \tag{4}$$

to non-relativistic quantum mechanical particles.

The non-relativistic paths thus look rather different from their relativistic counterparts: if $\Delta t = \varepsilon = (t_b - t_a)/N$ we have that $\Delta x \sim \sqrt{\Delta t} \sim \sqrt{1/N}$, so that for $N \to \infty$ the spatial steps are far larger than those in the relativistic case, where space and time steps scale in the same way.

In this paper we reconcile these results in the following way: although paths with all possible numbers of bends enter (2) the major contributors to the sum are those with very few bends (as we shall see more precisely below) and in fact successive steps are highly correlated in direction with one another. The correlation persists for roughly $1/(\varepsilon m)$ steps, which is to say that an actual reversal will typically occur only after a time $1/m$ $(= \varepsilon(1/\varepsilon m))$ in the rest frame of the particle. The number of reversals is therefore independent of the fineness (N) of the original division of the time interval. Thus if one wishes to describe a slowly moving particle by a sequence of collective steps of independent direction, the minimum collective step size (in space or in time) is of order $1/m$. This value, $1/m$, is the minimum Δx for which it is reasonable to assign to the particle a Markov process without memory. Moreover during such a collective step the velocity of the particle is 1, that is c, since it is taking elementary steps of equal size in space and time, all in the same direction. Thus for this interval,

$$\Delta x/\Delta t = 1.$$

Combining this with $\Delta x = 1/m$ we recover equation (4).

In the remainder of this paper we justify the foregoing picture, first by direct calculation of the 'typical' number of bends in a path and then by making use of a correspondence due to Gersch (1981) between the sum (2) and the one-dimensional Ising model.

2. Calculation of the typical number of reversals

Our goal is to estimate the number of reversals of direction that occur for the bulk of the contributors to the sum in (2). Consider then a path with R bends that enters the sum K_{-+}. It leaves moving right and arrives moving left. It makes exactly $1+(R-1)/2$ turns to the left and $(R-1)/2$ turns to the right (see figure 1). An arbitrary path satisfying these conditions can be generated by sprinkling $(R-1)/2$ arrows at arbitrary positions on the lower right side of the rectangle, signifying left turns, and the same number of arrows on the lower left side, signifying right turns. Suppose a total of N steps are taken, P to the right and Q to the left. The net distance travelled is $b-a=(P-Q)\varepsilon \equiv M$. It follows that

$$P=\tfrac{1}{2}(N+M) \qquad Q=\tfrac{1}{2}(N-M) \qquad -N \leqslant (P-Q) = M \leqslant N = (P+Q). \tag{5}$$

The $(R-1)/2$ arbitrary left turns can be made at any of the P right-moving steps except the (last) one where the particle reaches the upper right side of the rectangle (where it makes a compulsory left turn). The $(R-1)/2$ right turns also may appear

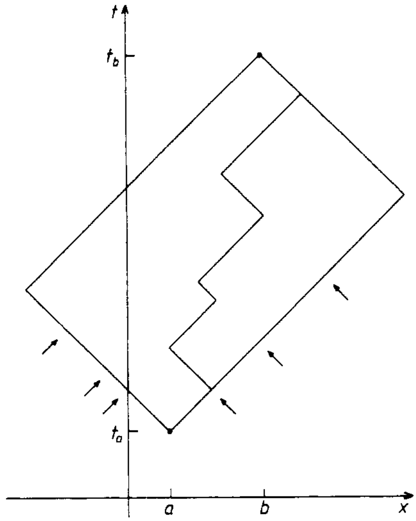

Figure 1. Space–time diagram for a path with seven bends that starts moving right and arrives moving left. A total of N steps are taken, P to the right, Q to the left. The net distance covered is $(P-Q)\varepsilon$ and the time elapsed is $(P+Q)\varepsilon$. With seven bends, there are three arbitrarily located left turns, three arbitrarily located right turns and a compulsory left turn just before arrival.

at any left-moving step but the last. Therefore

$$\Phi_{-+}(R)=\binom{P-1}{\frac{1}{2}(R-1)}\binom{Q-1}{\frac{1}{2}(R-1)} \tag{6}$$

since aside from the constraints just mentioned there is a one-to-one correspondence between paths and sprinklings of arrows. Clearly R is odd for Φ_{-+} and

$$0\leq\tfrac{1}{2}(R-1)\leq\min(P-1,Q-1).$$

In the limit $N\to\infty$, only values of R for which $R/N\to 0$ will contribute to the sum (2). To see this, suppose $R/N=\lambda$ for some fixed $\lambda>0$. Since $\Phi_{\beta\alpha}(R)<2^N$ we have

$$\Phi_{\beta\alpha}(R)(\varepsilon m)^R<2^N(\varepsilon m)^{\lambda N}=[2(m(t_b-t_a)/N)^\lambda]^N \tag{7}$$

and the last expression tends to zero faster than any power of N, whereas the terms summed below individually go as $1/N$. For $|b-a|<t_b-t_a$, P/N and Q/N remain finite in the limit so we also have $R/P\to 0$ and $R/Q\to 0$. (For $|b-a|=t_b-t_a$ no bends can occur and there is exactly one path. The propagator is therefore equal to unity on the light cone, which in the continuum limit coresponds to a δ-function singularity.)

When $|b-a|<t_b-t_a$ we may write

$$\binom{P-1}{\frac{1}{2}(R-1)}\approx\frac{P^{\frac{1}{2}(R-1)}}{\frac{1}{2}(R-1)!} \tag{8}$$

the equality being exact in the limit $N\to\infty$. A similar equation holds for Q from which, together with (8), (6) and (2), we obtain

$$K_{-+}=\sum_{\text{odd } R}(\mathrm{i}\varepsilon m)^R(PQ)^{\frac{1}{2}(R-1)}/[\tfrac{1}{2}(R-1)!]^2. \tag{9}$$

Now $PQ=\frac{1}{4}(N-M)(N+M)=(N/2\gamma)^2$, where $\gamma\equiv(1-v^2)^{-1/2}$ with $v^2=M^2/N^2=(b-a)^2/(t_b-t_a)^2$. We thus have

$$K_{-+}=\frac{2\gamma}{N}\sum_{\text{odd } R}\left(\frac{\mathrm{i}m(t_b-t_a)}{2\gamma}\right)^R\Big/\left[\left(\frac{R-1}{2}\right)!\right]^2. \tag{10}$$

This sum can be interpreted as a sum over alternatives (which is how Feynman presents the entire path integral formalism), each summand being the probability amplitude for paths having exactly R bends on the way from a to b. The most likely number of bends can then be evaluated by finding the summand of largest absolute value. To this end we let $m(t_b-t_a)/\gamma=z$ and write $(z/2)^R/[((R-1)/2)!]^2=\exp f(R)$. Then

$$\partial f/\partial R=\log\tfrac{1}{2}z-\log\tfrac{1}{2}(R-1).$$

The maximum probability occurs when $\partial f/\partial R=0$, i.e., at

$$R_0\sim z=[(t_b-t_a)/\gamma]mc^2/\hbar, \tag{11}$$

and the significant contributions come from R's satisfying

$$(R-R_0)^2\lessapprox|\partial^2 f/\partial R^2|_{R_0}^{-1}\sim R_0.$$

This confirms our original assertion that the number of bends does not increase with N and allows us to conclude that there is an average of one bend per unit time in the particle's rest frame, with time measured in Compton wavelengths (over c).

The sum (10) may be evaluated exactly in the limit $N\to\infty$. Writing $R=2k+1$ and letting the sum now go to $k=+\infty$, (10) becomes

$$(\mathrm{i}\varepsilon m)\sum_{k=0}^{\infty}(-1)^k(z/2)^{2k}/(k!)^2=\mathrm{i}\varepsilon m J_0(z) \tag{12}$$

where J_0 is the zeroth order Bessel function of the first kind (see Abramowitz and Stegun 1964). Since K_{-+} vanishes at every other lattice point it must be divided by 2ε to obtain the continuum form of the propagator. The other components of $K_{\beta\alpha}$ can be similarly evaluated with a small adjustment in the factors Φ_{++} and Φ_{--}:

$$\Phi_{++}(R)=\binom{P-2}{\frac{1}{2}R}\binom{Q-1}{\frac{1}{2}R-1}, \qquad \Phi_{--}=\Phi_{++}(P\leftrightarrow Q).$$

Following the same steps as before, one finds for all four components of the continuum propagator

$$K(x,t;0,0)=\frac{m}{2}\begin{pmatrix}(-t-x)J_1(m\tau)/\tau & \mathrm{i}J_0(m\tau)\\ \mathrm{i}J_0(m\tau) & (-t+x)J_1(m\tau)/\tau\end{pmatrix} \tag{13}$$

where $\tau\equiv(t^2-x^2)^{1/2}$ and $|x|<t$.

3. Equivalence to the Dirac equation and analogy with statistical mechanics

We next turn to a different method for evaluating the sum in (2), a method that will allow us to sharpen the ties to statistical mechanics (Gersch 1981). We shall see in what sense the relativistic stochastic process can be considered one in which the direction of successive steps is correlated and has correlation length $1/m$. To deal with the sum in (2) we define N variables σ_i, $i=1,\ldots,N$, each σ_i taking values $+1$ or -1, representing a spatial step to the right or left respectively on the ith time step. The indices α and β on $K_{\beta\alpha}$ are identified with σ_1 and σ_N. There is therefore a one-to-one correspondence between paths in the sum (2) and sequences $\{\sigma_i\}$ that satisfy $M \equiv \Sigma\, \sigma_i = (b-a)/\varepsilon$. The sum over all such possible sequences therefore provides the factor $\Phi_{\beta\alpha}(R)$. Thus

$$K_{\beta\alpha} = \sum_{\sigma_2=\pm1} \cdots \sum_{\sigma_{N-1}=\pm1} (\mathrm{i}\varepsilon m)^R \tag{14}$$

where only those sequences enter the sum for which $M = \Sigma\, \sigma_i$. Next note that R can be written in terms of the σ's as

$$R = \tfrac{1}{2} \sum_{i=1}^{N-1} (1 - \sigma_i \sigma_{i+1}). \tag{15}$$

Defining

$$\nu = -\tfrac{1}{2} \log(\mathrm{i}\varepsilon m) \tag{16}$$

it is clear that (14) is essentially the partition function for a one-dimensional Ising model with (coupling constant/temperature) $= \nu$, and the condition $M = \Sigma\, \sigma_i$ is interpreted as an evaluation of the partition function at fixed magnetisation, rather than the more common fixed external field condition. The usual form can be recovered by writing the constraint on the sum as a (Kronecker) delta

$$\delta\left(M, \sum \sigma_i\right) = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi} \exp\left[\mathrm{i}\theta\left(M - \sum \sigma_i\right)\right]. \tag{17}$$

Combining we obtain

$$K_{\beta\alpha} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}M\theta}}{2\pi} \sum_{\sigma_2 \ldots \sigma_{N-1}} \exp\left(\nu \sum_{i=1}^{N-1} \sigma_i \sigma_{i+1} - \mathrm{i}\theta \sum_{i=1}^{N} \sigma_i - (N-1)\nu \right) \tag{18}$$

where the sums over σ_i are unrestricted. As usual (Thompson 1972) we define the transfer matrix

$$L(\sigma, \sigma') = \exp[\nu\sigma\sigma' - \tfrac{1}{2}\mathrm{i}\theta(\sigma + \sigma') - \nu] \tag{19}$$

so that (18) becomes

$$K_{\beta\alpha} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}M\theta} (L^{N-1})_{\beta\alpha} \exp[-\tfrac{1}{2}\mathrm{i}\theta(\alpha + \beta)]. \tag{20}$$

The eigenvalues and eigenvectors of L are

$$\lambda_\pm = [\cos\theta \pm (\mathrm{e}^{-4\nu} - \sin^2\theta)^{1/2}] = [\cos\theta \pm \mathrm{i}(\sin^2\theta + \varepsilon^2 m^2)^{1/2}]$$
$$\Phi_\pm = \begin{pmatrix} \cos(\pm\frac{1}{4}\pi + \delta) \\ \sin(\pm\frac{1}{4}\pi + \delta) \end{pmatrix}, \qquad \tan 2\delta = \frac{\sin\theta}{\varepsilon m}. \tag{21}$$

By $P_\pm$ we mean the orthogonal projection operators $P_\pm = \Phi_\pm \Phi_\pm^\dagger$. From (21) it follows that these can be written

$$P_\pm = \tfrac{1}{2}[1 \pm (\sigma_x \cos 2\delta - \sigma_z \sin 2\delta)]. \tag{22}$$

In this notation $K_{\beta\alpha}$ becomes

$$K_{\beta\alpha}=\frac{1}{2\pi}\sum_{\mu=\pm1}\int_{-\pi}^{\pi}\mathrm{d}\theta\,(P_\mu)_{\beta\alpha}\exp[-\tfrac{1}{2}\mathrm{i}\theta\,(\alpha+\beta)]\,\mathrm{e}^{\mathrm{i}M\theta}$$
$$\times[\cos\theta+\mathrm{i}\mu(\varepsilon^2m^2+\sin^2\theta)^{1/2}]^{N-1}. \tag{23}$$

An interesting contrast to the Ising model emerges at this point since $|\lambda_+|=|\lambda_-|$ and neither dominates for large N. We shall see that the decay in correlations is a decay in phase rather than magnitude. For $N\to\infty$ the integral over θ may be evaluated through restriction to the region of stationary phase and to this end we consider the function

$$f(\theta)=\mathrm{i}M\theta+(N-1)\log[\cos\theta+\mathrm{i}\mu(\varepsilon^2m^2+\sin^2\theta)^{1/2}]. \tag{24}$$

Letting $v\equiv M/(N-1)$ the stationary phase condition $\partial f/\partial\theta=0$ yields

$$v=-\mu\sin\theta/(\varepsilon^2m^2+\sin^2\theta)^{1/2}\qquad\text{or}\quad\sin\theta=-\varepsilon mv\mu/(1-v^2)^{1/2}. \tag{25}$$

Calling the solution to (25) (that is nearer to zero) $\bar{\theta}$, we find

$$\left.\frac{\partial^2f}{\partial\theta^2}\right|_{\bar{\theta}}=-\mathrm{i}[(N-1)/\varepsilon m](1-v^2)[1-v^2(\varepsilon^2m^2+1)]^{1/2}. \tag{26}$$

Equation (26) indicates that the important contribution to the integral is in a range $|\theta|\lessdot 1/N$.

At this juncture two pleasing observations can be made. First, from (25) θ (or $\sin\theta$) is clearly the momentum conjugate to position (up to factors of ε). On the other hand, the Ising model context assigns to θ the role of field conjugate to magnetisation. The reason for this consistent parallelism is the common role of the Legendre transform. The second observation is that with (25) we can deduce fairly easily that K is indeed the propagator for the Dirac equation, (1). Considering only the solution of (25) for which $\theta=\mathrm{O}(\varepsilon)$ we write $p=\theta/\varepsilon$, drop $\mathrm{O}(\theta^2)$ terms as well as $\mathrm{O}(\varepsilon)$ terms not raised to high powers and bring K to the form

$$K_{\beta\alpha}(x,t)=\sum_{\mu=\pm1}\varepsilon\frac{1}{2\pi}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\mathrm{d}p\,\tfrac{1}{2}[1+\mu(\sigma_x\cos2\delta-\sigma_z\sin2\delta)]_{\beta\alpha}$$
$$\times\mathrm{e}^{\mathrm{i}px}[1+\mathrm{i}\mu\varepsilon(m^2+p^2)^{1/2}]^{N-1} \tag{27}$$

where $x=b-a$ and $t=t_b-t_a=N\varepsilon$. From (21) it follows that $\cos2\delta=m/(p^2+m^2)^{1/2}$ and $\sin2\delta=p/(p^2+m^2)^{1/2}$. Using $(1+A/N)^N\to\mathrm{e}^A$, (27) becomes

$$K_{\beta\alpha}(x,t)=\varepsilon\sum_{\mu=\pm1}\frac{1}{2\pi}\int\mathrm{d}p\frac{1}{2}\left[1+\mu\frac{(m\sigma_x-p\sigma_z)}{(p^2+m^2)^{1/2}}\right]_{\beta\alpha}\exp(\mathrm{i}px)\exp[\mathrm{i}\mu t(m^2+p^2)^{1/2}].$$

Finally the sum over μ can be done and the result combined in an exponential

$$K_{\beta\alpha}(x,t)=\varepsilon\frac{1}{2\pi}\int\mathrm{d}p\,\mathrm{e}^{\mathrm{i}px}[\exp\mathrm{i}t(m\sigma_x-p\sigma_z)]_{\beta\alpha}. \tag{28}$$

This is precisely the operator $\exp(-\mathrm{i}Ht)$, with H the Hamiltonian of (1), expressed as a Fourier transform. (The 'ε' appears because K as defined acts on the discrete spatial lattice, so the 'ε' provides the dx needed to use K as a continuous integral operator. Moreover, a second stationary point at (sign $\bar{\theta})\pi-\bar{\theta}$ provides a term equal in magnitude, but with a relative sign that takes care of the fact that the sum (2) gives a contribution only for M and N of the same parity. Thus the restriction '$N \pm M$ is even' actually implies $\varepsilon \to dx/2$, but the contribution of the second stationary point restores the factor 2.)

In the representation that we are using for the Dirac equation, upper and lower components of ψ correspond to amplitudes for velocity eigenstates. Thus a quantity that is suitable for indicating the persistence of direction for a slowly moving particle may be defined as follows. Let $F_T(t)$ be the probability amplitude for leaving the origin moving to the right at time 0, moving left at time t and returning to the origin moving right at time T. Thus

$$F_T(t) = \int \mathrm{d}y\, K_{+-}(0, T; y, t) K_{-+}(y, t; 0, 0),$$

where we sum over the intermediate spatial coordinate y, now taken as a continuous variable. Since it is a probability *amplitude* that we calculate, the analogy with statistical mechanics is not perfect; nevertheless we shall see that the concepts of long-range order, correlation length and near degeneracy of eigenvalues have relevance.

We recall (27) in the form

$$K_{-+} = \int \frac{\mathrm{d}p}{2\pi} \sum_{\mu} \tfrac{1}{2}[1 + \mu(\sigma_x \cos 2\delta - \sigma_z \sin 2\delta)]_{-+} \lambda_{\mu}^{N-1}\, \mathrm{e}^{\mathrm{i}px} \tag{29}$$

using the transfer matrix eigenvalues given in (21). For the matrices in (29) only σ_x has a non-zero $(-+)$ element. It follows that

$$K_{-+} = \frac{1}{2} \int \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px} \frac{m}{(m^2+p^2)^{1/2}} (\lambda_{+}^{N-1} - \lambda_{-}^{N-1}) = \mathrm{i} \int \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}px} \frac{m \sin[t(m^2+p^2)^{1/2}]}{(m^2+p^2)^{1/2}}$$

$$\begin{aligned} F_T(t) &= \frac{1}{16\pi^2} \int \mathrm{d}y\, \mathrm{d}p_1\, \mathrm{d}p_2 \exp[\mathrm{i}y(p_1 - p_2)][m/(p_1^2 + m^2)^{1/2}][m/(p_2^2+m^2)^{1/2}] \\ &\quad \times (\lambda_{+}^{N_2-1} - \lambda_{-}^{N_2-1})(\lambda_{+}^{N_1-1} - \lambda_{-}^{N_1-1}) \qquad (32a) \\ &= \frac{-1}{2\pi} \int \mathrm{d}p \frac{m^2}{(p^2+m^2)} \sin[(T-t)(m^2+p^2)^{1/2}] \sin[t(m^2+p^2)^{1/2}]. \qquad (32b) \end{aligned}$$

Using trigonometric identities in (32b) and differentiating with respect to t yields

$$\partial F_T(t)/\partial t = -m^2 \int \frac{\mathrm{d}p}{2\pi} \frac{\sin[(p^2+m^2)^{1/2}(T-2t)]}{(p^2+m^2)^{1/2}} = \mathrm{i}mK_{-+}(0, T-2t).$$

This can be integrated to obtain

$$F_T(t) = \mathrm{i}m \int_0^t K_{-+}(0, T-2s)\, \mathrm{d}s. \tag{33}$$

Now let us phrase the physical question. If $T \gg 1/m$ there will be many reversals of direction in the interval $[0, T]$. If we look at the initial behaviour with t then $F_T(t)$ will indicate the likelihood of the first switch in direction through its growth away from zero at time zero. Using (13) and the asymptotic (large T) form of the Bessel function we calculate the following

$$F_T(t) \sim \tfrac{1}{4} m(2/\pi mT)^{1/2} \{\sin[m(T-2t) - \tfrac{1}{4}\pi] - \sin(mT - \tfrac{1}{4}\pi)\} \qquad (34a)$$

$$K_{++}(0, T) \sim -\tfrac{1}{2} m(2/\pi mT)^{1/2} \sin(mT - \tfrac{1}{4}\pi). \qquad (34b)$$

For small t we see that indeed $F_T(t)$ is small and the ratio $F_T(t)/K_{++}(0, T)$ is first equal to unity when $2mt = \pi$ or $t = \pi/2m$, representing the first time at which the particle is sure to be moving left given that it starts and ends moving right at the origin at times 0 and $T(\gg t)$ respectively.

The role of eigenvalue degeneracy can be seen from (32). Near degeneracy for the eigenvalues (i.e., $\lambda_+^N - \lambda_-^N \sim 0$) within the important range of integration is sufficient to guarantee smallness of $F_T(t)$ for $t \ll 1/m$. This can also be seen directly from $(\lambda_+/\lambda_-)^N \sim [\exp(2im\varepsilon)]^N$.

The two eigenvalues $\lambda_\pm = 1 \pm i\varepsilon(m^2+p^2)^{1/2}$ specify the time dependence of negative and positive energy states (see (27) and the next equation), and it is the interference between these that leads to the phenomenon of *Zitterbewegung*. In our calculations ε is much shorter than $1/m$, so that the steps in the sum (2) are in fact at a level below the *Zitterbewegung*. The latter phenomenon is then expressed through the alternation of direction on the time scale $1/m$.

Although we have not discussed $\langle\sigma_k\rangle$, the expected direction of motion at any particular time, it is evident that it vanishes if and only if x does.

Therefore in the Ising model treatment of the Dirac particle we have the following rough correspondences: $x \leftrightarrow$ magnetisation M, $p \leftrightarrow$ external magnetic field h, $m \leftrightarrow$ temperature (more precisely $\varepsilon m \leftrightarrow e^{-2J/T}$, J = coupling constant, T = temperature), $1/m \leftrightarrow$ correlation length. (For the massless particle $|x| \neq t$ is impossible, which corresponds to having spontaneous magnetisation at the phase transition.)

One difference in the formalisms is that in statistical mechanics one is truly speaking of probabilities, and the correlation length refers to a *bona fide* conditional probability. Moreover, it is obtained from the ratio of the eigenvalues, a quantity whose magnitude differs from one and approaches one in the $T=0$, $h=0$ limit. For the Dirac particle $|\lambda_+| = |\lambda_-|$ and only the difference $\lambda_+^N - \lambda_-^N$ has a chance of vanishing. The correlation length therefore refers to a persistence of phase.

4. Concluding remarks

It is interesting to compare the situation discussed here with the case of Brownian motion. If one describes physical Brownian motion by a Markov process with $\Delta x^2/\Delta t = D$, where D is a fixed diffusion constant, the limit Δt, $\Delta x \to 0$ entails $\Delta x/\Delta t \to \infty$. From a physical point of view this limit is not relevant, since when Δt becomes shorter than the collision time, successive steps become correlated. This correlation is a consequence of the law of inertia.

In the quantum mechanical situation, we have seen that from the point of view of the relativistic treatment, the non-relativistic picture with 'imaginary diffusion constant' $i\hbar/m$ is invalid when Δt becomes shorter than $\hbar/mc^2$ since then successive steps are correlated. This transition occurs as the step speed reaches c, and the correlation follows from the correct relativistic rule that the amplitude for a sequence of steps is $(i\varepsilon m)^R$ with R the number of reversals of direction.

The parallel we have just drawn can be taken further in the context of gauge theories with spontaneously broken symmetry in which the mass of the electron arises from its interaction with the ground state of a Higgs field. In fact, the sum (2) can be thought of as a perturbation expansion, each switch in direction corresponding to scattering by an external potential $\langle\phi\rangle$. The time $1/f\langle\phi\rangle = 1/m$ then corresponds to a 'collision time', determined by the ground state expectation value $\langle\phi\rangle$ of the Higgs field and the strength f of the electron–Higgs coupling.

Although the origins of the electron mass remain an open question, we feel that some progress has been made in extending the analogy of quantum mechanics and Brownian motion to a lower microscopic level where random motion is no longer the dominant feature and a limiting velocity is present.

Acknowledgment

One of us (TJ) gratefully acknowledges the support provided by a United States–Israel Educational Foundation grant. This research was partially supported by NSF grant PHY81-07381.

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